Problem 3

There are \( n \geq 1 \) asynchronous processes, each with a private memory of \( p \) bits. The processes can communicate only through reading and writing a shared memory of \( s \) bits. Computation proceeds in a sequence of atomic steps: in each step, first a scheduler chooses one of the processes; then the chosen process reads and modifies his private state and the shared state. A *fair* scheduler does not neglect any process forever; a *bounded fair* scheduler does not neglect any process for more than \( b \) steps, for some integer bound \( b \geq n \).

You need to design a protocol that allows the processes to elect a leader. In addition to the private memory of \( p \) bits, each process \( i \) has a read-only variable \( F_i \) and a write-only variable \( V_i \). The variable \( F_i \in \{1, \ldots, n\} \) represents the initial favorite of process \( i \) for leader. The variable \( V_i \in \{1, \ldots, n\} \) represents the final vote of process \( i \) for leader. In addition to the shared memory of \( s \) bits, there is a shared write-only bit \( T \), which is initially 0. Setting \( T \) to 1 signals the completion of the election process. The requirements on the protocol are as follows:

1. Leader election must be unanimous, that is, when \( T = 1 \), then \( V_i = V_j \) for all \( 1 \leq i, j \leq n \).
2. One of the initial favorites must be elected leader, that is, when \( T = 1 \), then \( |\{j \mid F_j = V_1\}| \geq |\{j \mid F_j = i\}| \) for all \( 1 \leq i \leq n \).
3. Leader election must terminate, that is, if the scheduler is fair, then \( T = 1 \) within a finite number of steps.

Each process can initialize his private memory (with the exception of the \( F_i \)'s), but the initial state of the shared memory (with the exception of \( T \)) is unknown. Each process knows the number \( n \) of processes, but is oblivious to the choices made by the scheduler.

Questions:

1. What is the smallest number \( s \) of shared bits necessary to solve the problem?
2. Given a solution, for a scheduler with bound \( b \), let \( t_b \) be the maximal number of steps till termination (\( T = 1 \)), and let \( s \cdot t_b \) be the cost of the solution. What is the least cost necessary to solve the problem for bounded schedulers with unknown bound \( b \)?