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An Interface Theory for Relational Nets

Mini-project report for the Problem Solving in Computer Science course

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Abstract

We propose the assume/guarantee and port-dependency interface algebra (A/G-PD) as an interface theory for the relational nets algebra (RNA). We prove that A/G-PD is an interface algebra for RNA, but we disprove in Section 4.2.1 that A/G-PD is an interface theory for RNA.

1 Introduction

Relational nets (RNs), as described in [1], is a general model of components that lacks a suitable interface algebra. However, sub-classes of relational nets, namely rectangular nets and total nets, have suitable interface theories: the assume/guarantee (A/G) interface theory for rectangular nets, respectively the port-dependency (PD) interface theory for total nets. The paper [1] provides examples that show that neither of these two interface theories is a an interface theory for unrestricted relational nets. We noticed that the counterexample used for showing that A/G algebra is not an interface theory for relational nets is handled properly by a PD interface, and vice-versa. This suggests the idea of combining the (implementation) restricting power of these two interface theories into an A/G-PD interface algebra. This new interface algebra is also motivated by the intuitive feeling that the more expressive the components are, the more constraining the interfaces should be to handle them properly.

2 The stateless assume/guarantee and port-dependency interfaces

A *stateless A/G-PD interface* F consists of a stateless I/O interface $\Pi_F = (I_F, O_F^+, O_F)$, a satisfiable predicate ϕ_F on I_F called *input assumption*, a satisfiable predicate ψ_F on O_F called *output guarantee*, and an *I/O-dependency relation* $\kappa_F \subseteq I_F \times O_F$.

In other words, an A/G-PD interface combines all the components of a PD interface and of an A/G interface.

Composition $F \| G$ is defined iff $\Pi_F \| \Pi_G$ is defined. Then, $\phi_{F\|G} = \phi_F \wedge \phi_G$, and $\psi_{F\|G} = \psi_F \wedge \psi_G$, and $\kappa_{F\|G} = \kappa_F \cup \kappa_G$.

Connection $F\theta$ is defined iff (1) $\Pi_F\theta$ is defined, (2) the input assumption $\phi_{F\theta} = (\forall O_{F\theta})(\psi_F \wedge \rho_\theta \Rightarrow \phi_F)$ is satisfiable, and (3) for all channels $(x, y) \in \kappa_F$, we have $(y, x) \notin \theta$. Then, $\psi_{F\theta} = \psi_F \wedge \rho_\theta$, and $\kappa_{F\theta} = \kappa^*_{F\theta} \cap (I_{F\theta} \times O_{F\theta})$ where $\kappa^*_{F\theta}$ is defined as the smallest transitive relation on $P_{F\theta} \times O_{F\theta}$ such that $K_F \subseteq \kappa^*_{F\theta}$ and $\theta \subseteq \kappa^*_{F\theta}$.

Hierarchy $F' \preceq F$ iff (1) $\Pi_{F'} \preceq \Pi_F$, (2) $\phi_F \Rightarrow \phi_{F'}$, (3) $\psi_{F'} \Rightarrow \psi_F$, and (4) $\kappa_{F'} \cap (I_F \times O_F) \subseteq \kappa_F$.

Given a relational net f and a stateless A/G-PD interface F, we say $f \lhd_{A/G-PD} F$ iff (1) $f \lhd_{I/O} \Pi_F$, (2) $\kappa_f^* \cap (I_F \times O_F) \subseteq \kappa_F$, (3) $\phi_F \Rightarrow (\exists O_f)\rho_f$, and (4) $(\exists I_f)\rho_f \Rightarrow \psi_F$.

3 Each A-G/PD interface has an RN implementing it

Proof We present the way to construct a RN f from a given A-G/PD interface s.t. $f \lhd_{A/G-PD} F$. We are given the interface as a tuple: $F = (I_F, O_{F+}, O_F, \kappa_F, \phi_F, \psi_F)$.

We create a RN f, with:

- A_f containing two processes, that we name p_1 and p_2. Process p_1 will contain no output ports, and the input ports will be all the input ports in the input predicate ϕ_F. Process p_1 will have associated as predicate $\rho_{p_1} = \phi_F$.
  Process p_2 will contain no input ports, and the output ports will be chosen as a superset of O_F and a subset of O_F^+ all the input ports in the input predicate ϕ_F. Process p_2 will have associated as predicate $\rho_{p_2} = \psi_F$.
- $C_f = \emptyset$

We now prove that $f \lhd_{A/G-PD} F$:

- It is trivial that I_f is the set of input ports of p_1 and that $I_f \subseteq I_F$.
- It is trivial that O_f is the set of output ports of p_2 and, from the way we defined it, $O_F \subseteq O_f \subseteq O_F^+$
- $\kappa_f^* = \emptyset$. Therefore $\kappa_f^* \cap (I_F \times O_F) \subseteq \kappa_F$
- $\phi_F \Rightarrow (\exists O_f)\rho_f$ is true (where ρ_f is defined as in Section 4.2 of [1], so $\rho_f = \phi_F \wedge \psi_F$) because whenever ϕ_F is true, there exists a valuation of the ports in O_f that satisfies ρ_f (more exactly ψ_F, since ϕ_F is already satisfied), since ψ_F is satisfiable for some valuation of the output ports of .
- $(\exists I_f)\rho_f \Rightarrow \psi_F$ is true because if we have a valuation of the input ports that satisfies ρ_f, then it means that ϕ_F is satisfied. Because $\rho_f = \phi_F \wedge \psi_F$, ψ_F has to be true.

It is obvious that the construction of a RN can be applied on each interface. Therefore, each interface has a component that implements it. (QED)

Example Suppose we have the A/G-PD interface with the following components:

$I_F = \{i_1 : \mathbb{N}, i_2 : \mathbb{N}, i_3 : \mathbb{N}\}$
$O_F = \{o_1 : \mathbb{N}, o_2 : \mathbb{N}\}$
$O_F^+ = \{o_1 : \mathbb{N}, o_2 : \mathbb{N}\}$
$\kappa_F = \{(i_1, o_1), (i_2, o_2), (i_1, o_2), (i_3, o_2)\}$
$\phi_F = (i_1 < 100) \wedge (i_2 < 1000) \wedge (i_1 * i_2 = 100)$
$\psi_F = (o_1 = 0) \wedge (0_2 = 0)$

Then the RN f that we create with the procedure presented above is the one from Figure 1.

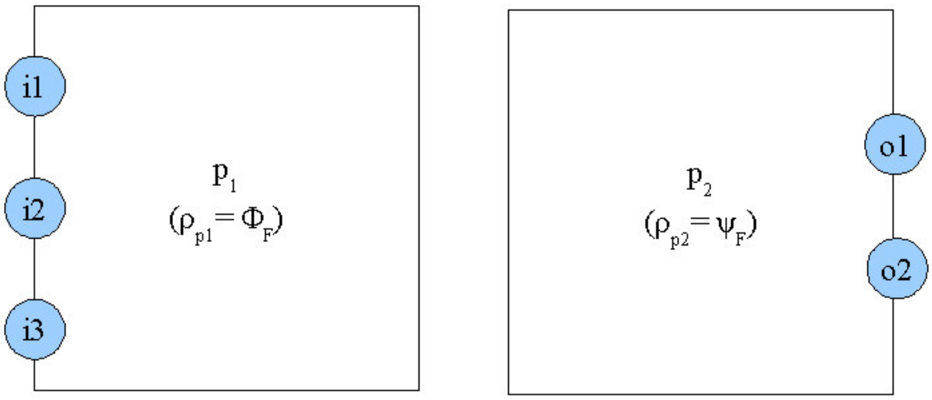

Figure 1. Component f implementing our example A/G-PD interface.

We now prove something outside the scope of our strict requirements of the project: each RN f has an A/G-PD interface F such that $f \lhd_{A/G-PD} F$.

Proof We are given f as a tuple (A_f, C_f) from which we can infer I_f and O_f. We choose $I_F = I_f$, $O_F = \emptyset$, and $O_F^+ = O_f$. Then we have to set $\kappa_F = \emptyset$ (since $\kappa_F \subseteq I_F \times O_F$, since $O_F = \emptyset$), $\psi_F = true$ and $\phi_F = false$.

Then $f \lhd_{A/G-PD} F$, because

- $\kappa_f^* \cap (I_F \times O_F) \subseteq \kappa_F$ is true because $O_F = \emptyset$ and $\kappa_F = \emptyset$.
- $\phi_F \Rightarrow (\exists O_f)\rho_f$, because false implies true or false $(\exists I_f)\rho_f \Rightarrow \psi_F$, because $\psi_F = true$

It is obvious that the constructive definition of an A-G/PD interface that is implemented by f is possible for any f. Therefore, each component has an interface that implements it. (QED)

4 $\lhd_{A/G-PD}$ is compositional

To prove that $\lhd_{A/G-PD}$ is compositional we have to prove that:

1. For all relational nets f and g, and all A/G-PD interfaces F and G, if $f \lhd_{A/G-PD} F$ and $g \lhd_{A/G-PD} G$ and $F\|G$ is defined, then $f\|g$ is defined and $f\|g \lhd_{A/G-PD} F\|G$.
2. For all relational nets f, all A/G-PD interfaces F, and all interconnects θ, if $f \lhd_{A/G-PD} F$ and $F\theta$ is defined, then $f\theta$ is defined and $f\theta \lhd_{A/G-PD} F\theta$.
3. For all relational nets f, and all A/G-PD interfaces F and F', if $f \lhd_{A/G-PD} F'$ and $F' \preceq F$, then $f \lhd_{A/G-PD} F$.

4.1 Implementation is compositional with respect to the composition operator

This proof is split into two proofs: a proof that $f\|g$ is defined and a proof that $f\|g \lhd_{A/G-PD} F\|G$.

4.1.1 Proof that $f\|g$ is defined

To prove that $f\|g$ is defined we have to prove that f and g have disjoint sets of ports.

Since $f \lhd_{I/O} F$ (from the definition of $\lhd_{A/G-PD}$) we have that $I_f \subseteq I_F$ and $O_f \subseteq O_F^+$. Since $P_f = I_f \cup O_f$ and $P_F^+ = I_F \cup O_F^+$ we have that $P_f \subseteq P_F^+$. For the same reasons we have that $P_g \subseteq P_G^+$. Moreover, since $F\|G$ is defined we have that $P_F^+ \cap P_G^+ = \emptyset$. Then we can conclude that $P_f \cap P_g = \emptyset$ which means that $f\|g$ is defined.

4.1.2 Proof that $f\|g \lhd_{A/G-PD} F\|G$

To prove that $f\|g \lhd_{A/G-PD} F\|G$ we have to prove the four points given in the definition of $\lhd_{A/G-PD}$.

Point 1. of $\lhd_{A/G-PD}$ definition: $f\|g \lhd_{I/O} \Pi_F\|\Pi_G$ To prove that we have to prove that $I_{F\|G} \subseteq I_{f\|g}$, $O_{f\|g} \subseteq O^+_{F\|G}$ and $O_{F\|G} \subseteq O_{f\|g}$.

From $f \lhd_{I/O} F$ we have that $I_F \subseteq I_f$, $O_f \subseteq O_F^+$ and $O_F \subseteq O_f$. The same things hold when replacing f by g and F by G. Since we have $I_{f\|g} = I_f \cup I_g$, $O_{f\|g} = O_f \cup O_g$, $O_{F\|G} = O_F \cup O_G$, $O^+_{F\|G} = O_F^+ \cup O_G^+$ and $O_{F\|G} = O_F \cup O_G$, it holds that $I_{F\|G} \subseteq I_{f\|g}$, $O_{f\|g} \subseteq O^+_{F\|G}$ and $O_{F\|G} \subseteq O_{f\|g}$. So $f\|g \lhd_{I/O} \Pi_F\|\Pi_G$.

Point 2. of $\lhd_{A/G-PD}$ definition: $\kappa^*_{f||g} \cap (I_{F||G} \times O_{F||G}) \subseteq \kappa_{F||G}$ We know that:

$$\begin{aligned}
\kappa^*_{f||g} &= (\textstyle\bigcup_{a\in A_{f||g}} I_a \times O_a \cup C_{f||g})^* \\
&= ((\textstyle\bigcup_{a\in A_f} I_a \times O_a \cup C_f) \cup (\textstyle\bigcup_{a\in A_g} I_a \times O_a \cup C_g))^*
\end{aligned}$$

and since we know that f and g have disjoint sets of ports we can write:

$$\begin{aligned}
\kappa^*_{f||g} &= ((\textstyle\bigcup_{a\in A_f} I_a \times O_a \cup C_f) \cup (\textstyle\bigcup_{a\in A_g} I_a \times O_a \cup C_g))^* \\
&= (\textstyle\bigcup_{a\in A_f} I_a \times O_a \cup C_f)^* \cup (\textstyle\bigcup_{a\in A_g} I_a \times O_a \cup C_g)^* \\
&= \kappa^*_f \cup \kappa^*_g
\end{aligned}$$

Thus we have the following equalities:

$$\begin{aligned}
\kappa^*_{f||g} \cap (I_{F||G} \times O_{F||G}) &= (\kappa^*_f \cup \kappa^*_g) \cap ((I_F \times O_F) \cup (I_F \times O_G) \cup (I_G \times O_F) \cup (I_G \times O_G)) \\
&= (\kappa^*_f \cap (I_F \times O_F)) \cup (\kappa^*_g \cap (I_G \times O_G))
\end{aligned}$$

All other intersections are empty because f and g have disjoint port sets. Then, since we know that $\kappa^*_f \cap (I_F \times O_F) \subseteq \kappa_F$, $\kappa^*_g \cap (I_G \times O_G) \subseteq \kappa_G$ and $\kappa_{F||G} = \kappa_F \cup \kappa_G$, we can conclude that $\kappa^*_{f||g} \cap (I_{F||G} \times O_{F||G}) \subseteq \kappa_{F||G}$.

Point 3. of $\lhd_{A/G-PD}$ definition: $\phi_{F||G} \Rightarrow (\exists O_{f||g})\rho_{f||g}$ We know that $\phi_F \Rightarrow (\exists O_f)\rho_f$ and $\phi_G \Rightarrow (\exists O_g)\rho_g$ and $\phi_{F||G} = \phi_F \wedge \phi_G$. Thus, we have that $\phi_{F||G} \Rightarrow ((\exists O_f)\rho_f \wedge (\exists O_g)\rho_g)$. Since we know that f and g have disjoint port sets, we can rewrite this as $\phi_{F||G} \Rightarrow ((\exists O_f \cup O_g)\rho_f \wedge \rho_g)$, which is exactly $\phi_{F||G} \Rightarrow (\exists O_{f||g})\rho_{f||g}$.

Point 4. of $\lhd_{A/G-PD}$ definition: $(\exists I_{f||g})\rho_{f||g} \Rightarrow \phi_{F||G}$ We know that $(\exists I_f)\rho_f \Rightarrow \psi_F$ and $(\exists I_g)\rho_g \Rightarrow \psi_G$ and $\psi_{F||G} = \psi_F \wedge \psi_G$ and $\rho_{f||g} = \rho_f \wedge \rho_g$. So we have that $(\exists I_f)\rho_f \wedge (\exists I_g)\rho_g \Rightarrow \phi_{F||G}$. Since we know that f and g have disjoint port sets, we have that $(\exists I_f)\rho_f \wedge (\exists I_g)\rho_g$ is equivalent to $(\exists I_f \cup I_g)\rho_f \wedge \rho_g$, which in turn is equivalent to $(\exists I_{f||g})\rho_{f||g}$. So we have $(\exists I_{f||g})\rho_{f||g} \Rightarrow \psi_{F||G}$.

4.2 Implementation is compositional with respect to the connection operator

We did not manage to prove this proposition completely and we put "Not done" for the parts that still need some more work.

4.2.1 Proof that $f\theta$ is defined - FALSE PROPOSITION

We will give a counterexample, inspired form [1], paragraph 4.3. Suppose we have two RNs, A and B. A consists of process a and B of process b, an no channels. Suppose that process a has a boolean output port x and the I/O relation x=0, and b has a boolean input port y and the I/O relation y=1.

We define interface F_A:

- $I_{F_A} = \emptyset$, $O_{F_A} = x$, $O^+_{F_A} = x$
- $\kappa_{F_A} = \emptyset$
- $\phi_{F_A} = true$, $\psi_{F_A} = true$

$A \lhd_{A/G-PD} F_A$ mainly because:

- $\phi_{F_A} \Rightarrow (\exists O_A)\rho_A$, because the implication $true \Rightarrow (\exists x)x = 0$ is true and
- $(\exists I_A)\rho_A \Rightarrow \psi_{F_A}$, because the implication $(\exists I_A)x = 0 \Rightarrow true$ is true, because an implication to true is always true.

We define interface F_B:

- $I_{F_B} = y$, $O_{F_B} = \emptyset$, $O^+_{F_B} = \emptyset$
- $\kappa_{F_B} = \emptyset$
- $\phi_{F_B} = true$, $\psi_{F_B} = true$

$B \lhd_{A/G-PD} F_B$ mainly because:

- $\phi_{F_B} \Rightarrow (\exists O_B)\rho_B$, because the implication $true \Rightarrow (\exists O_B)y = 1$ is true, because $O_B = \emptyset$ and $y = 1$ (from the predicate of B)
- $(\exists I_B)\rho_B \Rightarrow \psi_{F_B}$, because the implication $(\exists y)y = 1 \Rightarrow true$ is true.

It is obvious that $F_C = F_A || F_B$ is defined, with the following components:

- $I_{F_C} = y$, $O_{F_C} = x$, $O^+_{F_C} = x$
- $\kappa_{F_C} = \emptyset$
- $\phi_{F_C} = \phi_{F_A} \wedge \phi_{F_B} = true \wedge true = true$,
  $\psi_{F_C} = \psi_{F_A} \wedge \psi_{F_B} = true \wedge true = true$

We now try to connect the ports x and y of F_C with the connection $\theta = \{(x, y)\}$. $F_{C\theta}$, which we do not know yet if it is defined, has the following predicates:

- $I_{F_{C\theta}} = I_{F_C} \setminus O_\theta = \{y\} \setminus \{y\} = \emptyset$,
  $O_{F_{C\theta}} = O_{F_C} \cup O_\theta = \{x\} \cup \{y\} = \{x, y\}$
- $\phi_{F_{C\theta}} = (\forall O_{F_{C\theta}})(x = y \Rightarrow true) = (\forall x, y)true = true$ and
- $\psi_{F_{C\theta}} = true$

Since $\phi_{F_{C\theta}} = true$, $\phi_{F_{C\theta}}$ is satisfiable, so $F_{C\theta}$ is defined
But, obviously, $A||B$ is not defined.
Therefore, this proposition that we are trying to prove is false!!!!

Following is a sketch of the parts of the proof that we approached:

Proving that $f\theta$ is defined is equivalent to proving that $(A_{f\theta}, C_{f\theta})$ is a relational net, where $A_{f\theta} = A_f$ and $C_{f\theta} = C_f \cup \theta$.

Point 1. in relational net definition: all processes have disjoint port sets Trivial because f and $f\theta$ have the same processes and f is a relational net.

Point 2. in relational net definition: output ports of processes and target ports of channels must be disjoint sets Formally, we want to prove that:

$$\begin{array}{lrcll} & O_{A_{f\theta}} & \cap & O_{C_{f\theta}} & = \emptyset & \\ \Leftrightarrow & O_{A_f} & \cap & (O_{C_f} \cup O_\theta) & = \emptyset & \\ \Leftrightarrow & (O_{A_f} \cap O_{C_f}) & \cup & (O_{A_f} \cap O_\theta) & = \emptyset & (i) \end{array}$$

The left part of the disjunction (i) is empty because f is a relational net. For the right part we need to use the fact that $F\theta$ is defined which implies $O_\theta \cap O_F^+ = \emptyset$, and the fact that $f \lhd_{I/O} F$ (from the definition of $\lhd_{A/G-PD}$) which implies $O_f \subseteq O_F^+$.

More precisely we have: $O_{A_f} \subseteq O_f \subseteq O_F^+$ and $O_F^+ \cap O_\theta = \emptyset$ which implies that right part of the disjunction (i) must also be empty.

Point 3. in relational net definition: different channels must not have the same target From $F\theta$ defined we learn that channels in θ do not have the same target and, since f is a relational net, we also know that channels in C_f do not have the same target. It remains to prove that $O_{C_f} \cap O_\theta = \emptyset$. This is shown by exactly the same argument we use to prove that $O_{A_f} \cap O_\theta = \emptyset$ in the previous paragraph (use $O_{C_f} \subseteq O_f \subseteq O_F^+$ and $O_\theta \cap O_F^+ = \emptyset$).

Point 4. in relational net definition: existence of a consistent valuation IMPOSSIBLE TO PROVE - THE PROPOSITION SHOULD BE FALSE... !!!!

4.2.2 Proof that $f\theta \lhd_{A/G-PD} F\theta$ - FALSE PROPOSITION

Following is a sketch of the parts of the proof that we approached:

To prove this we first show that:

- $I_{f\theta} = I_f \setminus O_\theta$
  because, from the definition $I_{f\theta} = (\bigcup_{a\in A_f} I_a) \setminus O_{C_f} \setminus O_\theta$ and $I_f = (\bigcup_{a\in A_f} I_a) \setminus O_{C_f}$

- $O_{f\theta} = O_f \setminus O_\theta$ because, from the definition $O_{f\theta} = P_{f\theta} \setminus I_{f\theta}$ and $P_{f\theta} = I_f \cup O_f \cup I_\theta \cup O_\theta = I_f \cup O_f \cup O_\theta$, because $I_\theta \subseteq O_F \subseteq O_f$.
  $O_{f\theta} = (P_f \cup O_\theta)\setminus(I_f \setminus O_\theta = [(P_f \cup O_\theta)\setminus I_f]\cup[(P_f\cup O_\theta)\cap I_f \cap O_\theta]$ (because $A\setminus(B\setminus C) = (A\setminus B)\cup(A\cap B\cap C)) = (O_f\cup O_\theta)\cup(P_f\cap I_f\cap O_\theta)\cup(O_\theta\cap I_f\cap O_\theta) = (O_f\cup O_\theta)\cup(I_f\cap O_\theta)\cup(I_f\cap O_\theta) = (O_f\cup O_\theta)\cup(I_f\cap O_\theta) = O_f\cup O_\theta$

We now start proving the points of the implementation relation:

- $I_{f\theta} \subseteq I_{F\theta}$, because $I_{f\theta} = I_f \setminus O_\theta$, $I_f \subseteq I_F$ and $I_{F\theta} = I_F \setminus O_\theta$,
- $O_{F\theta} \subseteq O_{f\theta}$, because $O_{f\theta} = O_f \cup O_\theta$ and $O_{F\theta} = O_F \cup O_\theta$,
- $O_{f\theta} \subseteq O^+_{F\theta}$, because $O_{f\theta} = O_f \cup O_\theta$ and $O^+_{F\theta} = O^+_F \cup O_\theta$,
- $\kappa^*_{f\theta} \cap (I_{F\theta} \times O_{F\theta}) \subseteq K_{F\theta}$
- $\phi_{F\theta} \Rightarrow (\exists O_{f\theta})\rho_{f\theta}$ and
  $(\exists I_{f\theta})\rho_{f\theta} \Rightarrow \psi_{F\theta}$
  WE WERE NOT ABLE TO PROVE IT... !!!!

4.3 Implementation is compositional with respect to the hierarchy relation

We want to prove that if $f \lhd_{A/G-PD} F'$ and $F' \preceq F$ then $f \lhd_{A/G-PD} F$.

Point 1. of $\lhd_{A/G-PD}$ definition: $f \lhd_{I/O} \Pi_F$ We know that $f \lhd_{I/O} \Pi_{F'}$ and that $\Pi_{F'} \preceq \Pi_F$. This means we have $I_f \subseteq I_{F'}$, $O_f \subseteq O^+_{F'}$, $O_{F'} \subseteq O_f$, $I_{F'} \subseteq I_F$, $O^+_{F'} \subseteq O^+_F$ and $O_F \subseteq O_{F'}$. Thus we have $I_f \subseteq I_F$, $O_f \subseteq O^+_F$ and $O_F \subseteq O_f$, which is exactly $f \lhd_{I/O} F$.

Point 2. of $\lhd_{A/G-PD}$ definition We want to show that $\kappa^*_f \cap (I_F \times O_F) \subseteq \kappa_F$. First of all, observe that:

$$\begin{aligned}
\underbrace{\kappa^*_f \cap (I_F \times O_F)}_{C} &= \kappa^*_f \cap \{[(I_F \setminus I_{F'}) \times O_F] \cup (I_{F'} \times O_F)\} \\
&= \underbrace{\{\kappa^*_f \cap [(I_F \setminus I_{F'}) \times O_F]\}}_{l} \cup \underbrace{\{\kappa^*_f \cap (I_{F'} \times O_F)\}}_{r}
\end{aligned}$$

Then, the proof of $C \subseteq \kappa_F$ works by showing that $l = \emptyset$, and that $r \subseteq \kappa_F$.

Let's first show that $l = \emptyset$:

Since we know that $I_f \subseteq I_{F'} \subseteq I_F$, we have that $I_f \cap (I_F \setminus I_{F'}) = \emptyset$.

Since we also know that $O_f \subseteq O^+_{F'} \subseteq O^+_F$ and that $O^+_F \cap I_F = \emptyset$, we have that $O_f \cap (I_F \setminus I_{F'}) = \emptyset$.

Then, since $P_f = I_f \cup O_f$, we have that $P_f \cap (I_F \setminus I_{F'}) = \emptyset$, and since $\kappa^*_f \subseteq (P_f \times P_f)$, we have that $l = \emptyset$.

We now show that $r \subseteq \kappa_F$:

From $f \lhd_{A/G-PD} F'$, we know that $\kappa^*_f \cap (I_{F'} \times O_{F'}) \subseteq \kappa_{F'}$. Then we can write $\kappa^*_f \cap (I_{F'} \times O_{F'}) \cap (I_F \times O_F) \subseteq \kappa_{F'} \cap (I_F \times O_F)$. Now, observing that $I_{F'} \subseteq I_F$ and $O_F \subseteq O_{F'}$ we have that $(I_{F'} \times O_{F'}) \cap (I_F \times O_F) = I_{F'} \times O_F$, and thus $k \subseteq (\kappa_{F'} \cap (I_F \times O_F))$. From $F' \preceq F$, we know that $\kappa_{F'} \cap (I_F \times O_F) \subseteq \kappa_F$, and thus, by transitivity of set inclusion, we have $k \subseteq \kappa_F$.

OBS: At least this proof required the inclusion in the definition of the A/G-PD interfaces of the available ports set, O^+_F and of the fact that $O_f \subseteq O^+_F$. Without this fact we were not able to prove this proposition.

Point 3. of $\lhd_{A/G-PD}$ definition: $\phi_F \Rightarrow (\exists O_f)\rho_f$ We know that $\phi_F \Rightarrow \phi_{F'}$ and $\phi_{F'} \Rightarrow (\exists O_f)\rho_f$, thus by transitivity of the implication we have that $\phi_F \Rightarrow (\exists O_f)\rho_f$.

Point 4. of $\lhd_{A/G-PD}$ definition: $(\exists I_f)\rho_f \Rightarrow \psi_F$ We know that $(\exists I_f)\rho_f \Rightarrow \psi_{F'}$ and $\psi_{F'} \Rightarrow \psi_F$, thus by transitivity of the implication we have that $(\exists I_f)\rho_f \Rightarrow \psi_F$.

5 The stateless A/G-PD algebra is as expressive as the PD algebra

In this section we show that a *stateless A/G-PD interface* on the subset of *total nets* is as expressive as the port dependency algebra. Let $(A_1, \lhd_{A/G-PD})$ and $(A_2, \lhd_{PD})$ be two interface theories for the Total Nets B. By definition of *expressiveness*, the theory $(A_{A/G-PD}, \lhd_{A/G-PD})$ is as *expressive* as the theory $(A_{PD}, \lhd_{PD})$ if there is a function α from the interfaces of A_{PD} to the interfaces of $A_{A/G-PD}$ such that the following holds:

1. For all interfaces F and G of A_{PD}, if $F \| G$ is defined, then $\alpha(F) \| \alpha(G)$ is defined and is equal to $\alpha(F \| G)$.

2. For all interfaces F of A_{PD}, and all interconnects θ,if $F\theta$ is defined, then $\alpha(F)\theta$ is defined and is equal to $\alpha(F\theta)$.

3. For all interfaces F and F' of A_{PD}, if $F' \preceq F$, then $\alpha(F') \preceq \alpha(F)$.

4. For all interfaces F of A_{PD} and all components f of B, if $f \lhd_{PD} F$ then $f \lhd_{A/G-PD} \alpha(F)$.

For some interfaces F of A_{PD} and G of $A_{A/G-PD}$,let us define a function α from A_{PD} to $A_{A/G-PD}$ such that:

1. $\Pi_F = \Pi_G$ where $\Pi_F = (I_F, O_F^+, O_F)$ and $\Pi_G = (I_G, O_G^+, O_G)$ for input ports I, output ports O and possible output port assignment O^+.

2. $\kappa_F = \kappa_G$, where κ is the port dependency relationship.

3. $\rho_\theta = true$ for all interconnects θ on A/G-PD.

4. $\psi_G = true$ and $\phi_G = true$ for all output ports and input ports of G respectively.

5.1 Proof for $\alpha(F)\|\alpha(G)$ is defined and is equal to $\alpha(F\|G)$

Claim For all interfaces F and G of A_{PD}, if $F\|G$ is defined, then $\alpha(F)\|\alpha(G)$ is defined and is equal to $\alpha(F\|G)$.
We are given that $F\|G$ is defined, then from the definition of composition of PD interfaces we know $\Pi_F\|\Pi_G$ is defined.
To show $\alpha(F)\|\alpha(G)$ is defined, we have to show that $\Pi_{\alpha(F)}\|\Pi_{\alpha(G)}$ is defined.
From the definition of α:

$$\Pi_{\alpha(F)} = \Pi_F \quad and \quad \Pi_{\alpha(G)} = \Pi_G$$

So,$\Pi_F\|\Pi_G$ is defined implies $\Pi_{\alpha(F)}\|\Pi_{\alpha(G)}$ is defined. Hence,$\alpha(F)\|\alpha(G)$ is defined.

Again from the definition of α:

$$\begin{aligned} \Pi_{\alpha(F\|G)} &= \Pi_{F\|G} \\ &= \Pi_F \cup \Pi_G \qquad (1) \\ \Pi_{\alpha F\|\alpha G} &= \Pi_{\alpha F \cup \alpha G} \\ &= \Pi_F \cup \Pi_G \qquad (2) \end{aligned}$$

From equation (1) and (2) we see that $\Pi_{\alpha(F\|G)} = \Pi_{\alpha F\|\alpha G}$.

5.2 Proof of $\alpha(F)\theta$ is defined and is equal to $\alpha(F\theta)$

Claim For all interfaces F of A_{PD}, and all interconnects θ, if $F\theta$ is defined, then $\alpha(F)\theta$ is defined and is equal to $\alpha(F\theta)$.
From the definition of *connection* for PD interface algebra, if $F\theta$ is defined, then (1) $\Pi_F\theta$ is defined and (2) $\forall(x,y) \in \kappa_F$, $(y,x) \notin \theta$.

Let $\alpha(F) = F'$, where F' is an interface of A/G-PD algebra. $F'\theta$ is defined iff:

(a) $\Pi_{F'}\theta$ is defined

(b) $\phi_{F'\theta} = (\forall O_{F'\theta})(\psi_{F'} \wedge \rho_\theta \Rightarrow \phi_{F'})$ is satisfiable, and

(c) If $(x, y) \in \kappa_{F'}$, then $(y, x) \notin \theta$

5.2.1 Proof for $\Pi_{F'}\theta$ is defined

$\Pi_{F'}\theta$ is defined iff (1) $I_\theta \subseteq O_{F'}$, and (2) $O_\theta \wedge O^+_{F'} = \emptyset$. Since, $O_{F'} = O_F$, $I_\theta \subseteq O_F$ implies $I_\theta \subseteq O_{F'}$. Also, $O_\theta \wedge O^+_F = \emptyset$ implies $O_\theta \wedge O^+_{F'} = \emptyset$. Hence,$\Pi_{F'}\theta$ is defined.

5.2.2 Proof for $\phi_{F'\theta} = (\forall O_{F'\theta})(\psi_{F'} \wedge \rho_\theta \Rightarrow \phi_{F'})$ is satisfiable

From the definition of α, if $\alpha(F) = F'$, then $\psi_{F'} = true$, $\phi_{F'} = true$ and $\rho_\theta = true$. Substituting these values in equation:

$$\phi_{F'\theta} = (\forall O_{F'\theta})(\psi_{F'} \wedge \rho_\theta \Rightarrow \phi_{F'})$$

we get:

$$\phi_{F'\theta} = (\forall O_{F'\theta})(true \Rightarrow true)$$

which is always satisfiable.

5.2.3 Proof for $(y, x) \notin \theta$

From the definition of α, we know that $\kappa_{F'} = \kappa_F$. So, $(x, y) \in \kappa_{F'} \Rightarrow (x, y) \in \kappa_F$, which implies $(y, x) \notin \theta$.

5.3 Proof for $\alpha(F') \preceq \alpha(F)$

Claim For all interfaces F and F' of A_{PD}, if $F' \preceq F$, then $\alpha(F') \preceq \alpha(F)$. From the definition of *Hierarchy* for PD interfaces, if $F' \preceq F$, then:

$$\Pi_{F'} \preceq \Pi_F, and \tag{3}$$
$$\kappa_{F'} \cup (I_F \times O_F) \subseteq \kappa_F. \tag{4}$$

To show $\alpha(F') \preceq \alpha(F)$, we have to show:

(a) $\Pi_{\alpha(F')} \preceq \Pi_{\alpha(F)}$

(b) $\phi_{\alpha(F)} \Rightarrow \phi_{\alpha(F')}$

(c) $\psi_{\alpha(F)} \Rightarrow \psi_{\alpha(F')}$

(d) $\kappa_{alpha(F')} \cup (I_{\alpha(F)} \times O_{\alpha(F)}) \subseteq \kappa_{alpha(F)}$

Proof of (a) follows from definition of α and equation (3). Proof of (b) and (c) follows from the definition of α. Proof of (d) follows from definition of α and equation (4)

5.4 Proof for $f \lhd_{A/G-PD} \alpha(F)$

Claim For all interfaces F of A_{PD} and all components f of B, if $f \lhd_{PD} F$ then $f \lhd_{A/G-PD} \alpha(F)$. From the definition of *implementation* for PD interface algebra, if $f \lhd_{PD} F$, then:

$$f \lhd_{I/O} F \tag{5}$$
$$\kappa_f^* \cap (I_F \times O_F) \subseteq \kappa_F \tag{6}$$

To show $f \lhd_{A/G-PD} \alpha(F)$, we need to show:

(a) $f \lhd_{I/O} \Pi_{\alpha(F)}$

(b) $\kappa_f^* \cap (I_{\alpha(F)} \times O_{\alpha(F)}) \subseteq \kappa_{\alpha(F)}$

(c) $\phi_{\alpha(F)} \Rightarrow (\exists O_f)\rho_f$

(d) $(\exists I_f)\rho_f \Rightarrow \psi_{\alpha(F)}$

From the definition of function α (a),(b) and (c) trivially follows. Let us look at the proof of statement (d). From the definition of *Total Nets*, $(\forall I_a)(\exists O_a)rho_a$ is true. This means, there exists an input evaluation such that left hand side(LHS) of (d) is true. We know from the definition of α that $\psi_{\alpha(F)}$ is true. So, true on LHS of (d) always implies true on RHS.

6 Conclusion

We have proved that our A/G-PD interface algebra is an interface algebra and that any RN has at least one interface that it implements. We also proved that our interface algebra is as expressive as the PD interface theory on the total nets.

We disproved that A/G-PD is an interface theory for RNA in Section 4.2.1.

Our efforts in trying to prove that A/G-PD is a suitable interface theory for the entire RNs were not succesful. We could not anticipate we will get stuck in the proof for the connection part of the compositionality. We should have realised this is a weak point and experiment with more test cases before trying to prove the theorem - we took from the beginning the counterexample from [1], paragraph 4.2 and we saw that it is not violating our hypothesis, so we did not bother immediately after to try to adapt the counterexample from paragraph 4.3. It was a costly decision.

The entire proving process was very lengthy, with a lot of assumptions we had to take into consideration. We have left the partial proofs in the idea that they might be useful for similar projects.

We would like to think that our work will guide others that attack this problem in making a better choice for their interface algebra.

References

[1] Luca de Alfaro and Thomas A. Henzinger. Interface theories for component-based design. *Lecture Notes in Computer Science*, 2211:148–??, 2001.