Problem Solving in Computer Science: Week 5

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1 Writing a proof

We start by analysing the proof of the following theorem [1, p.28].

Theorem 1. *If there is a one-to-one function on a set A to a subset of a set B and there is also a one-to-one function on B to a subset of A, then A and B are equipollent.*

Proof. Suppose f is a one-to-one map of A into B and g is one-to-one on B to A, as shown in Figure 1.

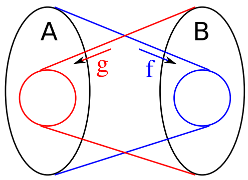

Figure 1: A diagram showing functions f, that is a one-to-one map of A into B, and g, that is a one-to-one map from B into A.

It may be supposed that $A \cap B = \emptyset$.

An element $x \in A \cup B$ is an ancestor of an element $y \in A \cup B$ iff y can be obtained from x by successive application of f and g (or g and f).

For an element $y \in A \cup B$, the ancestors of y are the *smallest* set $\mathrm{Anc}(y)$ such that $y \in \mathrm{Anc}(y)$ and for all $x, z \in A \cup B$, if $z \in \mathrm{Anc}(y)$ and $z = f(x)$ or $z = g(x)$, then $x \in \mathrm{Anc}(y)$. Figure 2 shows a chain representing the set $\mathrm{Anc}(y)$.

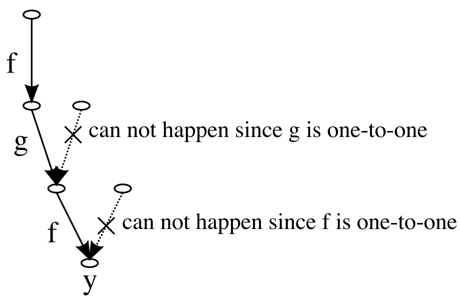

Figure 2: A chain representation of the ancestor set $\mathrm{Anc}(y)$.

Note that no node in the chain can have two direct predecessors, since both f and g are one-to-one functions.

Set A can be decomposed into three subsets, A_E, A_O and A_I, with the following properties:

$$\begin{array}{rl} x \in A_E, & \text{if } |\text{Anc}(x)| \text{ is even} \\ x \in A_O, & \text{if } |\text{Anc}(x)| \text{ is odd} \\ x \in A_I, & \text{if } |\text{Anc}(x)| \text{ is infinite} \end{array}$$

Note that, by their definition, sets A_E, A_O and A_I are pairwise disjoint.

The claim of the proof from [1, p.28] is that f maps A_E onto B_O and A_I onto B_I, and g^{-1} maps A_O onto B_E, as shown in Figure 3.

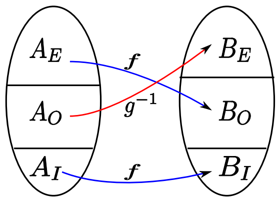

Figure 3: Constructing a bijection from set A to set B, as given in [1, p.28].

However, this claim is not true in cases when for some $x \in A \cup B$, the chain representing the set $\text{Anc}(x)$ contains a loop. A simple example of such an ancestor chain is shown in Figure 4, where both elements $x \in A$ and $y \in B$ have two, i.e. even number of ancestors, and contradict the previous claim.

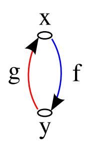

Figure 4: Looping ancestor chain.

The proof can be corrected by assigning the looping ancestor chains to sets A_I and B_I, thereby making sets A_E, A_O, B_E and B_O loop-free. In that case, there exists a bijection h from A to B, that agrees with f on $A_E \cup A_I$ and agrees with g^{-1} on A_O.

$\square$

The previous analysis showed how an omission of an important detail, which is not noticeable unless one really tries to formalize the proof, can make a proof incorrect. Therefore, before presenting a proof, it is a good exercise to first state it purely formally, as it gives a good starting point for identifying the important parts, which should be presented in the proof, and distinguish them from the trivial technicalities, which should be left out.

References

[1] John L. Kelley. *General Topology.* Springer-Verlag, New York, NY, USA, 1955.