Problem Solving in Computer Science

November 4, 2008

1 The Monkey Theorem

This is the solution to the homework that was given at the last class. Given the following two axioms:

Axiom 1 $(\forall x \mid monkey(x) \iff \neg human(x))$, which states that for all x, x is a monkey if and only if x is not a human;

Axiom 2 $(\forall x, y \mid anc(x,y) \iff x <_{par} y \vee (\exists z \mid anc(x,z) \wedge z <_{par} y))$, which states that for all x and y, x is an ancestor of y if and only if x is a parent of y or there exists z such that x is an ancestor of z and z is a parent of y.

Prove the following theorem, which states that if there is a human with an ancestor that is a monkey, then there must be a human with a parent that is a monkey:

Theorem 1 $\forall x, y\ (human(x) \wedge monkey(y) \wedge anc(y,x) \Rightarrow \exists u, v\ (human(u) \wedge monkey(v) \wedge v <_{par} u))$

Proof of Theorem 1

We want to show the statement of the theorem is true. In order to do this, we apply the method given in class. First, we identify the outermost symbol of the theorem: $\forall y$. We consider an arbitrary $\hat{y}$ and now have the following statement:

$\forall x\ (human(x) \wedge monkey(\hat{y}) \wedge anc(\hat{y},x) \Rightarrow \exists u, v\ (human(u) \wedge monkey(v) \wedge v <_{par} u))$

The outermost symbol is now $\forall x$, we therefore consider an arbitrary $\hat{x}$. As we intend to use the induction rule, we define the following induction hypothesis:

$\forall w, w <_{par} \hat{x} \Rightarrow (human(w) \wedge monkey(\hat{y}) \wedge anc(\hat{y},w) \Rightarrow \exists u, v\ (human(u) \wedge monkey(v) \wedge v <_{par} u))$

Note that the induction rule can be used because the relation $<_{par}$ is well-founded, *i.e.,* there is no infinite sequence $x_0 <_{par} x_1 <_{par} \ldots$.

We now have the following statement:

$human(\hat{x}) \wedge monkey(\hat{y}) \wedge anc(\hat{y},\hat{x}) \Rightarrow \exists u, v\ (human(u) \wedge monkey(v) \wedge v <_{par} u)$

The outermost symbol is: $\Rightarrow$. We now assume $human(\hat{x}) \wedge monkey(\hat{y}) \wedge anc(\hat{y},\hat{x})$, and show that $\exists u, v\ (human(u) \wedge monkey(v) \wedge v <_{par} u)$.

From Axiom 2, $\hat{y} <_{par} \hat{x} \vee (\exists z \mid anc(\hat{y},z) \wedge z <_{par} \hat{x})$.

Case 1: $\hat{y} <_{par} \hat{x}$ for $u = \hat{x}, v = \hat{y}$.

Case 2: Let $\hat{z}$ be such that $anc(\hat{y},\hat{z}) \wedge \hat{z} <_{par} \hat{x}$

From Axiom 1, we have $monkey(\hat{z}) \vee human(\hat{z})$

Case 1: $monkey(\hat{z})$, then $u = \hat{x}, v = \hat{z}$

Case 2: $human(\hat{z})$

By induction hypothesis, we know that $\hat{z} <_{par} \hat{x} \Rightarrow (human(\hat{z}) \wedge monkey(\hat{y}) \wedge anc(\hat{y}, \hat{z})) \Rightarrow \exists u, v \ (human(u) \wedge monkey(v) \wedge v <_{par} u))$.

2 Project 3

The third project will explore the construction of Binary Decision Diagrams (BDDs) and their storage in a pool with multiple concurrent access.

An introduction of BDDs was given in class and can be found in the class notes.