Problem Solving in Computer Science

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1 Important Informations

(1) Hand in report of project 2 by next Tuesday.
(2) Do Exercise 1 by next Tuesday.

2 Writing proofs (continued)

Outermost symbol	Goal	Assertion
$\forall$	In order to show $(\forall x \mid G(x))$, consider an arbitrary fixed $\hat{x}$ and show $G(\hat{x})$	If we know $(\forall x \mid A(x))$, we also know $A(e)$
$\forall$, $\prec$[1]	In order to show $(\forall x \mid G(x))$, consider an arbitrary fixed $\hat{x}$, assume $(\forall y \mid y \prec \hat{x} \Rightarrow G(y))$ and show $G(\hat{x})$ (proof by induction)	
$\exists$	In order to show $(\exists x \mid G(x))$, we show $G(e)$	If we know $(\exists x \mid A(x))$, let $\hat{x}$ be such that $A(\hat{x})$
$\Leftrightarrow$	In order to show $G \Leftrightarrow F$ (1) show $G \Rightarrow F$ (2) show $F \Rightarrow G$	If we know $A \Leftrightarrow B$, we can replace A by B in goals and assertions
$\Rightarrow$	In order to show $F \Rightarrow G$, assume F and show G	If we know $A \Rightarrow B$ then in order to show B it is sufficient to show A (and if we know A, we know B)
$\wedge$	In order to show $G \wedge F$ (1) show G (2) show F	If we know $A \wedge B$, we also know A and we know B
$\vee$	In order to show $G \vee F$, assume $\neg G$ and show F	If we know $A \vee B$, in order to show G, we consider two cases: (1) assume A, show G (2) assume B, show G
$\neg$	In order to show $\neg G$, assume G and show false (proof by contradiction).	But it is often better to move $\neg$ inside the goals and assertions (using de Morgan rules, for example)

2.1 Examples of well-founded relations

(1) $<$ on $\mathbb{N}$

(2) On binary trees:
$x \prec y$ if x is a proper subtree of y
$x \prec y$ if height of x less than height of y

[1] $\prec$ is a well-founded binary relation, i.e., there are no infinite sequences $x_0 \prec x_1 \prec x_2 \prec \ldots$

$x \prec y$ if x has fewer nodes than y

2.2 Proof by induction on natural numbers $\mathbb{N}$

Show $(\forall x \in \mathbb{N} | G(x))$
Consider arbitrary $\hat{x} \in \mathbb{N}$
Assume $(\forall y \in \mathbb{N} | y < \hat{x} \Rightarrow G(y))$, this is our induction hypothesis (IH)
To show $G(\hat{x})$
we know that $(\forall n \in \mathbb{N} | n = 0 \vee (\exists m \in \vee | n = m + 1))$
we also know that $\hat{x} = 0 \vee (\exists m \in \vee | \hat{x} = m + 1))$
case $\hat{x} = 0$, show $G(0)$
case $(\exists m \in \vee | n = m + 1))$
Let $\hat{m}$ be such that $\hat{x} = \hat{m} + 1$
Show $G(\hat{x})$, ie $G(\hat{m} + 1)$
By (IH) we know that $G(\hat{m})$ (inductive step)

3 Exercise 1

Symbols:
(1) monkey(x)
(2) human(x)
(3) $x \prec_{par} y$, meaning that x is a parent of y. Relation $\prec_{par}$ is well-founded.
Axiom 1: $(\forall x | \text{monkey}(x) \Leftrightarrow \neg\text{human}(x))$
Axiom 2: $(\forall x, y | \text{anc}(x, y) \Leftrightarrow x \prec_{par} y \vee (\exists z | \text{anc}(x, z) \wedge z \prec_{par} y))$

Prove that if there is a human with an ancestor that is a monkey, then there must be a human with a parent that is a monkey.

4 Exercise 2: Function unknown

Function unknown: strings $\rightarrow$ strings
unknown(x) = **if** $x = \epsilon$ (empty string) or $x = a$ (single character) **then** x
else first(unknown(rest(x))) $\cdot$ unknown(first(x) $\cdot$ unknown(rest(unknown(rest(x))))
with first$(a \cdot x) = a$ and rest$(a \cdot x) = x$ ($\forall$ string x, character a)

Axiom 1: reverse$(\epsilon) = \epsilon$
Axiom 2: $(\forall \text{character } a, \text{string } x | \text{reverse}(a \cdot x) = \text{reverse}(x \cdot a)$

Prove that $(\forall x|$unknown(x) = reverse(x))

First we need a well-founded relation $\prec$ on strings such that:
(1) rest(x) $\prec$ x
(2) rest(reverse(rest(x))) $\prec$ x
Let x $\prec$ y if length(x) < length(y).

Proof:

Consider arbitrary string $\hat{x}$
Assume ($\forall$ strings $y |$length$(y) <$ length$(\hat{x}) \Rightarrow$ unknown$(y) =$ reverse(y)) (IH)
Show unknown$(\hat{x})$ = reverse$(\hat{x})$

- case $\hat{x} = \epsilon$ show unknown(ϵ) = reverse$(\epsilon) = \epsilon$

- case $\hat{x} = \hat{a}$ show $\text{unknown}(\hat{a}) = \text{reverse}(\hat{a}) = \hat{a}$
- case $\hat{x} = \hat{a} \cdot \hat{y}$ for some character $\hat{a}$ and string $\hat{y}$. Show $\text{unknown}(\hat{a} \cdot \hat{y}) = \text{reverse}(\hat{a} \cdot \hat{y}) = \text{reverse}(\hat{y}) \cdot \hat{a}$