Markov Chains with Continuous Time

Transition Probabilities and Probability Distributions

Let $(X(t), t \geq 0)$ be a continuous-time Markov chain with state space S. For simplicity, we assume that S is finite and, w.l.o.g., $S = \{1, 2, \ldots, n\}$. For $i, j \in S$ and $h > 0$, let the transition probabilities

$$p_{ij}(h) = Pr\left(X(t+h) = j \mid X(t) = i\right)$$

be time-independent, that is, $(X(t), t \geq 0)$ is time-homogeneous. We define $T(h)$ as the matrix with entries $p_{ij}(h)$ and $T(0)$ as the identity matrix. Then, for all $t, h \geq 0$,

$$T(t+h) = T(t) \cdot T(h). \tag{1}$$

This equation is well-known as the *Chapmann-Kolmogorov Equation*.

Let us fix the initial distribution $\pi(0)$ of $(X(t), t \geq 0)$ and denote by $\pi(t)$ the probability distribution at time t, i.e., for the i-th entry of $\pi(t)$,

$$\pi_i(t) = Pr\left(X(t) = i\right).$$

Therefore,

$$\pi(t) = \pi(0) \cdot T(t).$$

The matrices $T(t)$ can be "generated" as follows. We define the *generator matrix*

$$Q = \lim_{h \to 0} \frac{1}{h}\left(T(h) - T(0)\right) = \frac{d}{dt} T(t)|_{t=0}. \tag{2}$$

of $(X(t), t \geq 0)$. Let $(q_{ij})_{i,j \in S}$ denote the entries of Q, also referred to as *rates*. The matrix Q has the following properties.

- For an infinitesimal time interval $[t, t+h)$ and $i, j \in S$ with $i \neq j$,

$$\begin{aligned} Pr\left(X(t+h) = j \mid X(t) = i\right) &= q_{ij} \cdot h, \\ Pr\left(X(t+h) = i \mid X(t) = i\right) &= 1 - \textstyle\sum_{j \neq i} q_{ij} \cdot h. \end{aligned}$$

- The row sums in Q are zero and the diagonal entries are non-positive, i.e., for all $i, j \in S$,

$$0 \geq q_{ii} = -\sum_{j \neq i} q_{ij}.$$

Combining (2) and (1) yields the *Kolmogorov forward and backward equations*

$$\begin{aligned} \tfrac{d}{dt} T(t) &= T(t) \cdot Q, \\ \tfrac{d}{dt} T(t) &= Q \cdot T(t) \end{aligned} \tag{3}$$

With initial condition $T(0) = I$, they have the solution

$$T(t) = e^{Qt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k, \tag{4}$$

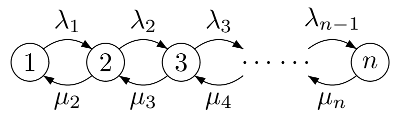

Figure 1: Intensity graph of a finite birth-death process.

which describes how the transition probabilities can be obtained from Q. Summarizing,

$$\pi(t) = \pi(0) \cdot T(t) = \pi(0) \cdot e^{Qt}. \tag{5}$$

In general, the computation of the matrix exponential is difficult if n is large. Most approaches for the computation of $\pi(t)$ are based on an approximate solution of the differential equation

$$\frac{d}{dt}\pi(t) = \pi(t) \cdot Q. \tag{6}$$

Componentwise, (6) states that

$$\begin{aligned} \tfrac{d}{dt}\pi_i(t) &= \textstyle\sum_{j\neq i} q_{ji}\pi_j(t) + q_{ii}\pi_i(t) \\ &= \textstyle\sum_{j\neq i} q_{ji}\pi_j(t) - \sum_{j\neq i} q_{ij}\pi_i(t) \end{aligned} \tag{7}$$

The intuitive interpretation is that the change of $\pi_i(t)$ is given by the probability mass that enters state i minus the mass that leaves i.

From (5) is follows that $(X(t), t \geq 0)$ is uniquely determined by Q and $\pi(0)$. Thus, we can represent $(X(t), t \geq 0)$ as a state-transition graph, often referred to as the *intensity graph* of $(X(t), t \geq 0)$. For each entry $q_{ij} > 0$, there is a transition with label q_{ij} from i to j. The diagonal entry q_{ii} is then the negative sum of all labels of transitions emanating from i. We call $(X(t), t \geq 0)$ *ergodic* if its intensity graph is strongly connected.

Example: A finite *birth-death process* is a Markov chain with a generator matrix of the form

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & \cdots & 0 \\ \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots & 0 \\ 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ 0 & \cdots & \cdots & 0 & \mu_n & -\mu_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R}_{>0}$ are the *birth rates* and $\mu_2, \ldots, \mu_n \in \mathbb{R}_{>0}$ are the *death rates*. The associated intensity graph is depicted in Figure 1. By (7),

$$\tfrac{d}{dt}\pi_i(t) \quad = \quad \lambda_{i-1}\pi_{i-1}(t) + \mu_{i+1}\pi_{i+1}(t) - \lambda_i\pi_i(t) - \mu_i\pi_i(t).$$

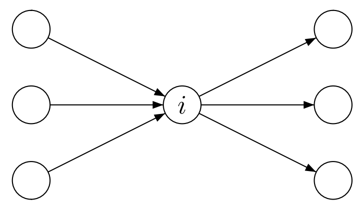

Figure 2: The global balance equations.

For a finite, continuous-time Markov chain $(X(t), t \geq 0)$, the limiting distribution

$$\pi^* = \lim_{t \to \infty} \pi(t)$$

always exists (in contrast to discrete-time Markov chains). If $(X(t), t \geq 0)$ is ergodic, for all $t \geq 0$,

$$\pi^* \cdot T(t) = \pi^*,$$

and it can be shown that π^* is the unique solution of the linear equation system

$$\pi^* \cdot Q = 0, \quad \sum_{i=1}^{n} \pi_i^* = 1. \tag{8}$$

These equations are called *global balance equations* and π^* is often referred to as *steady-state distribution*, because π^* is independent of $\pi(0)$. Intuitively, the global balance equation

$$\textstyle\sum_{j \neq i} q_{ji} \pi_j^* \quad = \quad \textstyle\sum_{j \neq i} q_{ij} \pi_i^* \tag{9}$$

of state i expresses that, when the system is in equilibrium, the probability mass that enters i equals the mass that leaves i, as illustrated in Figure 2. If the intensity graph of $(X(t), t \geq 0)$ is not strongly connected, i.e., $(X(t), t \geq 0)$ is not ergodic, we can calculate the limiting distribution by identifying all strongly connected components (SCCs) in the graph. For each SCC, we can solve (8) where Q is the generator matrix of the SCC. The long-run probability of a state in an SCC is then given by the product of its steady-state probability in its SCC and the probability to enter its SCC in the limit. For all states i that do not belong to an SCC, $\pi_i^* = 0$. Note that the probability to enter an SCC in the limit depends on the initial distribution $\pi(0)$. Thus, in the non-ergodic case π^* depends on $\pi(0)$.