15 Project 3: Formalizing Riddles

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1 Introduction to riddles

Enigime des 5×5 is a widely known riddle, also often called "Einstein's Puzzle" or the "Zebra Puzzle." The riddle is about five people from different nations, who own five different pets, smoke five different cigarattes, live in five differently colored houses and drink five different beverages (hence the name 5×5). The hints to the riddle are given as a set of associations between these 25 elements. The riddle has a unique solution.

The complexity of the above riddle motivates the study of complexity of riddles. We generalize a riddle to be an $k \times n$ matrix, and a set h of hints.

2 Solving riddles as a SAT problem: A Boolean approach

Given a riddle as a $k \times n$ matrix, it can be encoded as a SAT problem with n^2k Boolean variables. We denote the Boolean variables as a_{ijl}, where $1 \leq i \leq k$ (attribute), $1 \leq j \leq n$ (position), and $1 \leq l \leq n$ (value). The intended meaning is: $a_{ijl} = \texttt{true}$ iff the (i, j) entry in the matrix has value l. The Boolean variables satisfy the following constraints:

- Every entry in the matrix has a value between 1 and n.

$$\bigwedge_{1 \leq i \leq k} \bigwedge_{1 \leq j \leq n} \bigvee_{1 \leq l \leq n} a_{ijl}$$

- Every entry in the matrix has atmost one value.

$$\bigwedge_{1 \leq i \leq k} \bigwedge_{1 \leq j \leq n} \bigwedge_{1 \leq l \leq n} \bigwedge_{\substack{1 \leq l' \leq n \\ l' \neq l}} \neg(a_{ijl} \wedge a_{ijl'})$$

- In every row, every column entry has a distinct value.

$$\bigwedge_{1 \leq i \leq k} \bigwedge_{1 \leq j \leq n} \bigwedge_{\substack{1 \leq j' \leq n \\ j' \neq j}} \bigwedge_{1 \leq l \leq n} \neg(a_{ijl} \wedge a_{ij'l})$$

Apart from these general constraints, the hints h in the riddle provide additional constraints. For example, in the riddle *Enigme des* 5×5, the first hint is equivalent to a_{111} assuming that the country row is 1 and the value of Norway is 1. The second hint is equivalent to $(a_{112} \wedge a_{211}) \vee (a_{122} \wedge a_{221}) \vee (a_{132} \wedge a_{231}) \vee (a_{142} \wedge a_{241}) \vee (a_{152} \wedge a_{251})$, assuming that the color row is 2, the value of English is 2, and the value of red is 1.

After encoding all the hints, we find out whether the conjunction is satisfiable. The satisfying assignment gives the solution to the riddle. A riddle is *uniquely solvable* if there exists exactly one satisfying assignment for the variables.

3 Solving riddles as a constraint satisfaction problem: A first-order approach

We describe a riddle in a language with predicate symbols $a[i, j] = l$, Boolean operators $\vee, \wedge, \neg$, and $\rightarrow$, and quantifiers $\forall$ and $\exists$ over rows, columns, and values.

All hints are given as constraints with at most one variable. Example of a zero-variable constraint: $a[1, 1] = 1$ (Hint 1). Example of a one-variable constraint:

$$(\,\forall j.1 \leq j \leq n\,)\, a[2, j] = 1 \rightarrow a[2, j + 1] = 2 \quad \text{(Hint 3).}$$

We note that the size of input description is $k \cdot n^3$ in the Boolean approach, and $O(1)$ in the first-order approach.

We would like to address the following questions.

(i) How does the problem specification change if all hints have to be specifiable as sentences of constant size in first order logic for hints?

(ii) When are two riddles different? Are there several non symmetrical 5x5 riddles, which are different from the riddle solved in class?